# Problem Set III, Solutions

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**Problem 0.1.** Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in $V$. Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in $W$.

*Proof.* Suppose for some $a_1, \ldots, a_n \in F$ that

$$a_1(Tv_1) + \cdots + a_n(Tv_n) = 0. \tag{0.1}$$

By the linearity of $T$,

$$a_1(Tv_1) + \cdots + a_n(Tv_n) = T(a_1v_1) + \cdots + T(a_nv_n) = T(a_1v_1 + \cdots + a_nv_n) = 0. \tag{0.2}$$

Since $T$ is injective, $\text{null}(T) = \{0\}$, so $a_1v_1 + \cdots + a_nv_n = 0$. Since $(v_1, \ldots, v_n)$ is linearly independent in $V$, we necessarily have that $a_1 = \cdots = a_n = 0$, so $(Tv_1, \ldots, Tv_n)$ is linearly independent as desired. □

**Problem 0.2.** Suppose that $p \in \mathcal{P}(\mathbb{C})$ has degree $m$. Prove that $p$ has $m$ distinct roots if and only if $p$ and its derivative $p'$ have no roots in common.

*Proof.* We first suppose that $p$ has $m$ distinct roots. As the degree of $p$ is $m$, we write

$$p(z) = \alpha(z - \lambda_1) \cdots (z - \lambda_m), \tag{0.3}$$

with $\alpha \in \mathbb{C}$ and $\lambda_1, \ldots, \lambda_m$ distinct.

Fix $i \in \{1, \ldots, m\}$. We will prove that $p'(\lambda_i) \neq 0$ for each $i$, thereby establishing that $p, p'$ have no common roots. We write $p$ as

$$p(z) = (z - \lambda_i)q(z) \tag{0.4}$$

such that $q(\lambda_i) \in \mathcal{P}(\mathbb{C})$ is nonzero. Differentiating yields

$$p'(z) = (z - \lambda_i)q'(z) + q(z) \Rightarrow p'(\lambda_i) = q(\lambda_i) \neq 0. \tag{0.5}$$

To prove the reverse direction, we show that if $p$ has less than $m$ distinct roots, then $p$ and $p'$ share at least one root. Suppose $p$ has less than $m$ distinct roots. Then for $d \geq 2$ and $q(z) \in \mathcal{P}(\mathbb{C})$ we may write

$$p(z) = (z - \lambda)^d q(z). \tag{0.6}$$

Differentiating yields

$$p'(z) = (z - \lambda)^d q' + d(z - \lambda)^{d-1} q(z). \tag{0.7}$$

We conclude that $p'(\lambda) = 0$, so $\lambda$ is our shared root. □